

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
MOCK EXAM

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ABSTRACT. This is the exam from Matthias Ruf's preceding course. Try to solve it within **120 minutes without any additional aid**. This mock exam makes no claim to completeness or comparability with respect to the selection of topics, level of difficulty, or types of questions in relation to the actual exam.

Spoiler alert. The exam starts on the next page. You may not want to read through the problems before you start the exam in order to avoid bias.

Problem 1 (True or false questions). For each question decide whether it is always true or if it is false in general, that means there are counterexamples to the claim. In the first case mark TRUE, while in the second case mark FALSE. Read the questions carefully in order to not miss a word!

- Let $f: B_1(0) \setminus \{0\} \rightarrow \mathbf{C}$ be a holomorphic function and assume that 0 is an essential singularity of f . Then either f attains all values in $\mathbf{C}$ infinitely many often or f is not surjective.
- There exists a sequence $(E_n)_{n \in \mathbf{N}}$ of holomorphic functions $E_n: \mathbf{C} \rightarrow \mathbf{C} \setminus \{0\}$ such that the infinite product $\prod_{n=1}^{\infty} (z - 1/n) E_n(z)$ converges locally normally on $\mathbf{C}$.
- Let $U \subset \mathbf{C}$ be an open set. If there exists a biholomorphic map $f: \mathbf{C} \rightarrow U$, then $U = \mathbf{C}$.
- Let $n \geq 2$ and $S \subset \mathbf{C}^n$ be a finite set. Then every holomorphic function $f: \mathbf{C}^n \setminus S \rightarrow \mathbf{C}$ can be extended to a holomorphic function $f: \mathbf{C}^n \rightarrow \mathbf{C}$.
- Let $D \subset \mathbf{C}$ be a domain and $(f_n)_{n \in \mathbf{N}}$ be a sequence of holomorphic functions $f_n: D \rightarrow \mathbf{C}$ that is locally uniformly bounded. Then the following statements are equivalent.
 - The sequence $(f_n)_{n \in \mathbf{N}}$ converges locally uniformly to some holomorphic function.
 - There exists $z_0 \in D$ such that for all $k \in \mathbf{N}_0$ the sequences $(f_n^{(k)}(z_0))_{n \in \mathbf{N}}$ converge.
- There exists a holomorphic function $f: \mathbf{C} \setminus \mathbf{N} \rightarrow \mathbf{C}$ such that at each even natural number the function f has an essential singularity, while at each odd natural number of the form $2k + 1$ it has a pole of order k .
- An infinite product $\prod_{n=1}^{\infty} a_n$ is absolutely convergent if and only if it holds that $\limsup_{n \rightarrow \infty} n^2 |a_n - 1| < \infty$.
- Let $G \subsetneq \mathbf{C}$ be a nonempty simply connected domain. Then there exists a biholomorphic map $f: B_1(0) \rightarrow G$ such that $f'(0) = 1$.

Problem 2. a. Find the largest open set $U \subset \mathbf{C}$ such that the infinite product

$$\prod_{n=1}^{\infty} \left[1 + \sin^2 \left[\frac{z}{n} \right] \right]$$

converges locally normally on U . Justify your answer.

- Let $(a_n)_{n \in \mathbf{N}}$ be a sequence of nonzero complex numbers in $B_1(0) \setminus \{0\}$ such that $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$. Show that the infinite product

$$F(z) := \prod_{n=1}^{\infty} \frac{\overline{a_n}}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

converges locally normally on $B_1(0)$ and defines a holomorphic function $F: B_1(0) \rightarrow \mathbf{C}$. Further justify that F vanishes exactly on the set $\{a_n : n \in \mathbf{N}\}$.

Problem 3. Let $G \subsetneq \mathbf{C}$ be a nonempty, simply connected domain and $f, g: G \rightarrow G$ be two biholomorphic maps. Assume that there exist nonequal $z_1, z_2 \in G$ with $f(z_1) = g(z_1)$ and $f(z_2) = g(z_2)$. Show that $f = g$ ¹

Problem 4. Let $f: \mathbf{C} \rightarrow \mathbf{C}$ be a nonconstant holomorphic function. Show that for every $z_0 \in \mathbf{C} \setminus \{0\}$ the equation $\exp(f(z)) = z_0$ has infinitely many solutions $z \in \mathbf{C}$.

¹**Reminder.** For all $z_0 \in B_1(0)$ the map $\varphi_{z_0}(z) := (z - z_0)/(1 - \overline{z_0}z)$ is a biholomorphic map from $B_1(0)$ to itself that maps z_0 to 0.

Problem 5. Let $U \subset \mathbf{C}$ be an open set and $(f_n)_{n \in \mathbf{N}}$ be a sequence of holomorphic functions $f_n: U \rightarrow \mathbf{C} \setminus \{0, 1\}$. Assume that for every $z \in U$ the sequence $(f_n(z))_{n \in \mathbf{N}}$ is bounded.

- a. Show that there exists a subsequence $(f_{n_k})_{k \in \mathbf{N}}$ that converges locally uniformly to a holomorphic function $f: U \rightarrow \mathbf{C}$.
- b. Give an example of an open set U and a sequence $(f_n)_{n \in \mathbf{N}}$ of holomorphic functions $f_n: U \rightarrow \mathbf{C} \setminus \{0, 1\}$ that converges locally uniformly on U to a function f that attains the value 0. Justify the properties.

Problem 6. Let $n \geq 2$ and let $K \subset \mathbf{C}^n$ be a compact set such that $\mathbf{C}^n \setminus K$ is connected. Let $f: \mathbf{C}^n \setminus K \rightarrow \mathbf{C}$ be holomorphic and bounded. Show f is constant.